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Generation of jets on K3 surfaces

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Abstract

Let L be an ample line bundle on a K3 surface X. We give sharp bounds on n such that the global sections of nL simultaneously generate k-jets on X. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Consider a K3 surface X and an ample line bundle L on X. It was established by Saint-Donat [7] – and follows also from Reider's theorem [6] – that $\mathcal{O}_X(2L)$ is globally generated and $\mathcal{O}_X(3L)$ is very ample. The purpose of this note is to see how these basic facts generalize to the generation of jets and to jet ampleness. Recall that L is said to *generate k-jets* at a point $x \in X$, if L has global sections with arbitrarily prescribed k-jets at x, i.e. if the evaluation map

$$H^0(X,L) \to H^0(X,L \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$$

is surjective. A stronger variant of this local notion includes the separation of a finite set of distinct points: L is *k-jet ample*, if for any choice of distinct points $x_1,\dots,x_r$ in

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X and positive integers $k_1,\dots,k_r$ with $\sum_{i=1}^{r} k_i = k+1$ the natural map

$$H^0(X,L) \to H^0(X, L \otimes \mathcal{O}_X/(\mathfrak{m}_{x_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{x_r}^{k_r}))$$

is surjective (see [1]). This means that L has global sections generating all simultaneous k-jets at any r points $x_1,\dots,x_r$.

Suitably high multiples of L will certainly separate any given number of points and jets, so the interesting problem here is to determine optimal bounds. Certain effective statements on the local generation of jets, which are however not sharp, can be obtained by considering the Seshadri constant $\varepsilon(L,x)$, which measures the local positivity of L at x. In fact, elementary arguments yield bounds for $\varepsilon(L,x)$ which, via vanishing, imply that the line bundle $\mathcal{O}_X(nL)$ generates k-jets at x for $n \geq k+2$ if L is globally generated, and for $n \geq 2k+4$ if the linear system $|L|$ has base points (see Section 3). Our main result gives the optimal bounds in this situation and also the sharp bound for jet ampleness:

Theorem. *Let X be a K3 surface, L an ample line bundle on X and k a non-negative integer. Then either*

(a) *$\mathcal{O}_X(nL)$ is k-jet ample for $n \geq k+2$, or*

(b) *L is of the form $L = \mathcal{O}_X(aE+\Gamma)$, where $E \subset X$ is an elliptic curve, $\Gamma \subset X$ is a (-2)-curve with $E \cdot \Gamma = 1$ and $a \geq 3$.*

In the exceptional case (b) *let Δ be the finite set of singular points of the fibres of the elliptic fibration $X \to \mathbb{P}^1$ given by $|E|$. Then $\mathcal{O}_X(nL)$ generates k-jets at a point $x \in X - \Delta$ for $n \geq k+2$ and it generates k-jets at a point $x \in \Delta$ if and only if $n \geq 2k+1$.*

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Notation and conventions. We work throughout over the field $\mathbb{C}$ of complex numbers and use standard notation in Algebraic Geometry.

For a $\mathbb{Q}$-divisor D we denote by $\lceil D \rceil$ its round-up and by $\lfloor D \rfloor$ its round-down (integer part). Line bundles and divisors are used with no distinction, as well as the additive and multiplicative notation.

We will make use of the Kawamata–Viehweg vanishing theorem, which states that for a nef and big $\mathbb{Q}$-divisor D on a smooth projective surface S one has

$$H^i(S, \mathcal{O}_S(K_S + \lceil D \rceil)) = 0 \quad \text{for } i > 0.$$

Note that there is no normal crossing hypothesis in the surface case of Kawamata–Viehweg vanishing thanks to Sakai's lemma (see [3, Lemma 1.1].

Let X be a non-singular surface. By a *jet* of order k (or a *k-jet*) of a linear system $|L|$ on X at the point $x \in X$ we mean an element $j \in H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$. Moreover, by a *simultaneous jet* of order k (or a *simultaneous k-jet*) at the points $x_1,\dots,x_r \in X$

we mean an element

$$j \in H^0(X, L \otimes \mathcal{O}_X/(\mathfrak{m}_{x_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{x_r}^{k_r})) = \bigoplus_{i=1}^{r} H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i}),$$

where $\sum_{i=1}^{r} k_i = k+1$. Given a global section $s \in H^0(X,L)$, we consider the local Taylor expansions of s around the x_i's. A simultaneous k-jet is then given by the $\sum_{i=1}^{r} \binom{k_i+1}{2}$-tuple of the coefficients of the terms of degree $\le k_i$ for each x_i. In the proof of Lemma 2.1 we will use the term 0-jet of order $k_i - 1$ at a point x_i for the $\binom{k_i+1}{2}$-tuple

$$\underline{0} = (0, \ldots, 0) \in H^0(X, L \otimes \mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i}) = \mathbb{C}^{\oplus \binom{k_i+1}{2}}.$$

1. Linear systems with base points

Let X be a K3 surface and L an ample line bundle on X. We are interested in the jet ampleness of tensor powers $\mathcal{O}_X(nL)$, $n \ge 1$. In this section we study the case where the linear system $|L|$ has base points. Under this assumption L is of the form

$$L = \mathcal{O}_X(aE + \Gamma), \tag{1}$$

where $E \subset X$ is an elliptic curve, $\Gamma \subset X$ is a smooth rational curve with $E \cdot \Gamma = 1$ and $a \ge 3$ (see [7, Proposition 8.1]). The (-2)-curve Γ is the base locus of $|L|$. The pencil $|E|$ gives an elliptic fibration $X \to \mathbb{P}^1$. For $x \in X$ we will denote by E_x the unique fibre passing through x. Because of $L \cdot E = 1$, all the fibres E_x are irreducible. The singular fibres E_x are rational curves with a single double point, which must lie outside Γ. Further, we will consider the finite set

$$\Delta =_{\text{def}} \{x \in X \mid E_x \text{ is singular at } x\}$$

of the singular points of the fibres.

First, we show:

Proposition 1.1. *Suppose that L is of the form* (1) *and let $x \in \Delta$. Then $\mathcal{O}_X(nL)$ generates k-jets at x for $n \ge 2k+1$.*

Proof. It is enough to show that

$$H^1(\tilde{X}, nf^*L - (k+1)Z) = 0 \quad \text{for } n \ge 2k+1, \tag{2}$$

where $f : \tilde{X} \to X$ is the blow-up of X in x and $Z = f^{-1}(x)$ is the exceptional divisor. Consider the divisor $D =_{\text{def}} aE_x + \Gamma \in |L|$ and the $\mathbb{Q}$-divisor

$$M =_{\text{def}} nf^*L - \left(k + \frac{1}{2}\right) Z - \lambda \left(nf^*D - \left(k + \frac{1}{2}\right) Z\right),$$

where λ is defined as

$$\lambda =_{\rm def} \frac{3}{n(4a-1)}.$$

Since $\mathrm{mult}_x(E_x)=2$, one easily checks that the $\mathbb{Q}$-divisor $f^*L - \frac{1}{2}Z$ is nef and big, hence the numerical equivalence

$$M \equiv (1-\lambda)\left(nf^*L - \left(k+\frac{1}{2}\right)Z\right)$$

shows that M is nef and big for $n \geq 2k+1$ as well. We will show that its round-up leads to the asserted vanishing.

Since $\mathrm{mult}_x(D)=2a$, we can write

$$f^*D = aE'_x + \Gamma' + 2aZ,$$

where E'_x and Γ' are the proper transforms of the curves E_x respectively Γ. Therefore, the round-up of M is

$$\lceil M \rceil = nf^*L - \lfloor \lambda na \rfloor E'_x - \lfloor \lambda n \rfloor \Gamma' - \left\lfloor 2\lambda na + (1-\lambda)\left(k+\frac{1}{2}\right)\right\rfloor Z.$$

Our choice of λ implies $\lambda na < 1$ and

$$2\lambda na + (1-\lambda)\left(k+\frac{1}{2}\right) \geq k+2,$$

hence we find

$$K_{\tilde{X}} + \lceil M \rceil = \lceil M \rceil + Z = nf^*L - (k+1)Z - pZ$$

for some $p \geq 0$. The Kawamata–Viehweg vanishing theorem then gives (2), which in turn shows that $\mathcal{O}_X(nL)$ generates k-jets at x. □

Now we prove that the bound $2k+1$ in the previous proposition is, in fact, sharp:

Proposition 1.2. *Let L be of the form* (1) *and $x \in \Delta$. Then $\mathcal{O}_X(nL)$ does not generate k-jets at x if $n \leq 2k$.*

Proof. First note that it is enough to prove the assertion for $n=2k$, since L is globally generated at x. Again let $f : \tilde{X} \to X$ be the blow-up at x, Z the exceptional divisor and $D =_{\rm def} aE_x + \Gamma$.

From the exact sequence

$$0 \to \mathcal{O}_X(2kL) \otimes \mathfrak{m}_x^{k+1} \to \mathcal{O}_X(2kL) \to \mathcal{O}_X(2kL) \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1} \to 0$$

and $H^1(X, 2kL)=0$ we see that it is sufficient to show that

$$H^1(\tilde{X}, 2kf^*L - (k+1)Z) \neq 0. \tag{3}$$

Define

$$\lambda =_{\mathrm{def}} \frac{2}{k(4a-1)}$$

and consider the $\mathbb{Q}$-divisor

$$M =_{\mathrm{def}} 2kf^*L - kZ - \lambda(2kf^*D - kZ).$$

Because $\lambda < 1$, M is certainly nef and big. Its round-up is

$$\lceil M \rceil = 2kf^*L - \lfloor 2k\lambda a \rfloor E'_x - \lfloor 2k\lambda \rfloor \Gamma' - \lfloor k + 4k\lambda a - k\lambda \rfloor Z,$$

where as before E'_x and Γ' are the proper transforms. The main point in the construction of M is that we have

$$1 < 2k\lambda a < 2, \quad 0 < 2k\lambda < 1 \quad \text{and} \quad k + 4k\lambda a - k\lambda = k + 2,$$

hence

$$\lceil M \rceil = 2kf^*L - E'_x - (k+2)Z,$$

which by Kawamata–Viehweg gives the vanishing

$$H^i(\tilde{X}, 2kf^*L - E'_x - (k+1)Z) = 0 \quad \text{for } i > 0. \tag{4}$$

The curve E'_x is the normalization of an irreducible singular elliptic curve, so it is smooth and rational. Consider the exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{\tilde{X}}(2kf^*L - E'_x - (k+1)Z \to \mathcal{O}_{\tilde{X}}(2kf^*L - (k+1)Z) \\ &\to \mathcal{O}_{E'_x}(2kf^*L - (k+1)Z) \to 0. \end{aligned}$$

From its associated long cohomology sequence and (4) we get

$$H^1(\tilde{X}, 2kf^*L - (k+1)Z) \cong H^1(E'_x, 2kf^*L - (k+1)Z|_{E'_x}). \tag{5}$$

But the restriction of $2kf^*L - (k+1)Z$ to E'_x is of degree -2, hence the right hand cohomology group in (5) does not vanish. This gives (3) and hence proves the proposition. □

Next we show:

Proposition 1.3. *Suppose that L is of the form* (1)*. Then $\mathcal{O}_X(nL)$ generates k-jets at points $x \in X - \Delta$ for $n \geq k+2$.*

Proof. Let x be a point in $X - \Delta$ and let k be a non-negative integer. We again denote by $f : \tilde{X} \to X$ the blow-up of X in x and by Z the corresponding exceptional divisor. Suppose $n \geq k+2$. To prove the proposition, it is enough to show that

$$H^1(\tilde{X}, nf^*L - (k+1)Z) = 0. \tag{6}$$

First, we have

$$(nf^*L-(k+2)Z)^2=n^2L^2-(k+2)^2\geq(k+2)^2L^2-(k+2)^2>0,$$

since the intersection pairing on X is even. Further, if $C'\subset\tilde{X}$ is any irreducible curve different from the exceptional divisors, then we can write $C'=f^*C-mZ$, where C is an irreducible curve on X and $m=\operatorname{mult}_x(C)$. We have

$$(nf^*L-(k+2)Z)\cdot C'=nL\cdot C-(k+2)m.$$

If C is the base curve Γ or the fibre E_x, then $x\notin\Delta$ implies $m\leq 1$, so

$$nL\cdot C-(k+2)m\geq n-(k+2)\geq 0.$$

If C is not as just described, the divisor

$$D=_{\text{def}}(k+1)(aE_x+\Gamma)\equiv(k+1)L$$

meets C properly, so

$$nL\cdot C\geq D\cdot C\geq a(k+1)m+(k+1)\Gamma\cdot C\geq(k+2)m.$$

So we have shown that $nf^*L-(k+2)Z$ is nef and big, hence (6) follows from the Kawamata–Viehweg vanishing theorem, and we are done. $\square$

Remark 1.4. The bound $k+2$ in the previous proposition is actually sharp. To see this, assume that $\mathcal{O}_X(nL)$ generates k-jets on $X-\Delta$ for some $n\leq k+1$ and consider the restriction of the bundle $\mathcal{O}_X(nL)$ to a smooth elliptic fibre E. The restriction of L to E is of the form $\mathcal{O}_E(p)$ for some point $p\in E$. Then $|\mathcal{O}_E(nL)|=|\mathcal{O}_E(np)|$ only generates $(n-2)$-jets at p, since there is no meromorphic function on E with a simple pole at p. A fortiori $|\mathcal{O}_X(nL)|$ does not generate $(n-1)$-jets at p.

2. Globally generated bundles

If a globally generated line bundle on a K3 surface fails to be very ample, then it gives a double covering of $\mathbb{P}^2$ or of a rational normal scroll. Therefore we begin this section by studying the generation of simultaneous k-jets in the set-up of a double covering.

Lemma 2.1. *Let $\pi:X\to Y$ be a double covering of smooth projective surfaces, branched over a smooth divisor $B\subset Y$. Suppose that $M\in\operatorname{Pic}(Y)$ is a k-jet ample line bundle and that $\mathcal{O}_Y(M-\frac{1}{2}B)$ is $(k-1)$-jet ample. Then $L=\pi^*M$ is k-jet ample.*

Proof. Let $R=\pi^*(B)_{\text{red}}$ and let $s_R\in H^0(R)$ be a section whose divisor of zeros is R. Observe that $s_R(-x)=-s_R(x)$ for all $x\in X$. The projection formula gives the following isomorphism

$$H^0(X,\pi^*M)\simeq \pi^*H^0(Y,M)\oplus s_R\cdot \pi^*H^0\left(Y,M-\frac{1}{2}B\right). \tag{7}$$

Now, let points $x_1,\dots,x_r\in X$ and positive integers $k_1,\dots,k_r$ with $\sum_{i=1}^r k_i=k+1$ be given. Furthermore, let

$$J\in H^0\left(X,\pi^*M\otimes \mathcal{O}_X\Big/\bigotimes_{i=1}^r \mathfrak{m}_{x_i}^{k_i}\right)=\bigoplus_{i=1}^r H^0(X,\pi^*M\otimes\mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i})$$

be given. Let us write, corresponding to the above sum decomposition, $J=(j_1,\dots,j_r)$, where j_i is a (k_i-1)-jet of $|\pi^*M|$ at x_i for $i=1,\dots,r$. We can then write $J=\sum_{i=1}^r J_i$ where the simultaneous k-jets J_i are of the form $J_i=(\underline{0},\dots,\underline{0},j_i,\underline{0},\dots,\underline{0})$. In other words J_i has the 0-jet of order k_l-1 as lth component, for $l\neq i$, and j_i as the ith component. It is enough to find for $i=1,\dots,r$ a section s_i whose simultaneous k-jet at the points $x_1,\dots,x_r$ is given by J_i, since the sum $s=\sum_{i=1}^r s_i$ will then have the prescribed simultaneous jet J. In order to alleviate notation, we assume $i=1$.

We distinguish between three cases.

Case 1: Suppose that $x_1\notin R$ and that x_2 is the second point in the fibre of π over $y_1=\pi(x_1)$.

Let p,q be local coordinates at the point $y_1\in Y$. The pull-back of these coordinates gives rise to local coordinates u_j,v_j around the points x_j for $j=1,2$. In these local coordinates j_1 can be written as

$$j_1=\sum_{i+j<k_1} a_{ij}u_1^iv_1^j$$

(since we can set $a_{ij}=0$ for $i+j\geq k_1$).

Let $\ell=\max(k_1,k_2)$. Since M and $\mathcal{O}_Y(M-\frac{1}{2}B)$ are $(k-1)$-jet ample and $\ell+\sum_{i\geq 3}k_i\leq k$, there are sections $s\in H^0(Y,M)$ and $t\in H^0(Y,M-\frac{1}{2}B)$ satisfying the following conditions:

- $s \bmod \mathfrak{m}_{y_1}^{\ell}=\frac{1}{2}\sum_{i+j<\ell}a_{ij}p^iq^j$,
- $s \bmod \mathfrak{m}_{\pi(x_i)}^{k_i}=\underline{0}$ for $i\geq 3$,
- $(s_R\cdot\pi^*t) \bmod \mathfrak{m}_{x_1}^{\ell}=\frac{1}{2}\sum_{i+j<\ell}a_{ij}u_1^iv_1^j$,
- $t \bmod \mathfrak{m}_{\pi(x_i)}^{k_i}=\underline{0}$ for $i\geq 3$.

Then we have

$$(s_R\cdot\pi^*t) \bmod \mathfrak{m}_{x_2}^{\ell}=-\frac{1}{2}\sum_{i+j<\ell}a_{ij}u_2^iv_2^j,$$

which in turn implies that

$$(\pi^*s+s_R\cdot\pi^*t) \bmod \mathfrak{m}_{x_1}^{\ell}=j_1,$$
$$(\pi^*s+s_R\cdot\pi^*t) \bmod \mathfrak{m}_{x_i}^{k_i}=\underline{0}\quad \text{for } i\geq 2.$$

Then the section $\pi^*s+s_R\cdot\pi^*t=s_1$ has, in fact, the prescribed jet J_1.

Case 2: Suppose that $x_1 \notin R$ and that the other point in the fibre of π over $y_1 = \pi(x_1)$ is not among $x_2, \ldots, x_r$. Then, keeping the notation from the preceding case, the k-jet ampleness of M implies that there exists a section $s \in H^0(Y, M)$ such that

- $s \bmod \mathfrak{m}_{y_1}^{k_1} = \sum_{i+j<k_1} a_{ij} p^i q^j$,
- $s \bmod \mathfrak{m}_{\pi(x_i)}^{k_i} = \underline{0}$.

Now $\pi^* s := s_1$ has the prescribed jet J_1.

Case 3: We assume now that $x_1 \in R$. Since B is smooth there are local coordinates (p, q) at the point $y_1 = \pi(x_1)$ such that $B = \{p = 0\}$. The local coordinates (u, v) around the point x_1 can be chosen in such a way that locally around x_1

$$\pi : (u, v) \to (p = u^2, q = v).$$

So we have $R = \{u = 0\}$ locally around the point x_1. We can write the jet j_1 in the following way:

$$\begin{aligned} j_1 &= \sum_{2i+j<k_1} a_{2i,j} u^{2i} v^j + \sum_{2i+1+j<k_1} a_{2i+1,j} u^{2i+1} v^j \\ &= \sum_{2i+j<k_1} a_{2i,j} u^{2i} v^j + u \cdot \sum_{2i+j<k_1-1} a_{2i+1,j} u^{2i} v^j. \end{aligned}$$

Since M is k-jet ample there exists a section $s \in H^0(Y, M)$ satisfying

- $s \bmod \mathfrak{m}_{y_1}^{k_1} = \sum_{2i+j<k_1} a_{2i,j} p^i q^j$,
- $s \bmod \mathfrak{m}_{\pi(x_i)}^{k_i} = \underline{0}$ for $i \geq 2$.

Similarly, since $\mathcal{O}_Y(M - \frac{1}{2}B)$ is $(k-1)$-jet ample there exists a section $t \in H^0(Y, M - \frac{1}{2}B)$ such that

- $t \bmod \mathfrak{m}_{y_1}^{k_1-1} = \sum_{2i+j<k_1-1} a_{2i+1,j} p^i q^j$,
- $t \bmod \mathfrak{m}_{\pi(x_i)}^{k_i} = \underline{0}$ for $i \geq 2$.

It is now easy to check that $\pi^* s + s_R \pi^* t$ has the prescribed jets at the points $x_1, \ldots, x_r$. $\square$

We now apply the lemma to show the following

Proposition 2.2. *Let X be a $K3$ surface and L be an ample globally generated line bundle on X. Then $\mathcal{O}_X(nL)$ is k-jet ample for $n \geq k + 2$.*

Proof. If L is already very ample, then clearly $\mathcal{O}_X(nL)$ is n-jet ample and we are done (see [1, Corollary 2.1]). On the other hand, if L fails to be very ample, then we are in one of the following two cases:

Case 1: $L^2 = 2$. Then Riemann–Roch implies $h^0(X, L) = 3$ and L induces a $2:1$ mapping $\pi : X \to \mathbb{P}^2$, which is branched over a smooth sextic $B \subset \mathbb{P}^2$. Setting $M = \mathcal{O}_{\mathbb{P}^2}(n)$ we have $nL = \pi^* M$ and

$$\mathcal{O}_{\mathbb{P}^2}\left(M - \frac{1}{2}B\right) = \mathcal{O}_{\mathbb{P}^2}(n) \otimes \mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}_{\mathbb{P}^2}(n-3),$$

hence the assumptions of the lemma are satisfied for $n \geq k + 2$ and we are done.

Case 2: $L^2 \geq 4$. Then Theorem 5.2 in [7] implies that there are two possibilities: either

(i) there exists a genus 2 curve $C \subset X$ such that $L = \mathcal{O}_X(2C)$, or

(ii) there exists an elliptic curve $E \subset X$ with $L \cdot E = 2$.

In the first case, since C is irreducible and since there are no linear systems on K3 surfaces having isolated base points, $\mathcal{O}_X(C)$ is globally generated. From Case 1 it follows that $\mathcal{O}_X(nC)$ is k-jet ample for $n \geq k+2$, and $\mathcal{O}_X(nL) = \mathcal{O}_X(2nC)$ is, in this case, even $(2k+2)$-jet ample.

In the second case, since L is ample, Proposition 5.7 of [7] implies that L gives a $2:1$ mapping

$$\pi : X \to \pi(X) \subset \mathbb{P}^{p_a(L)},$$

where $\pi(X)$ is a rational normal scroll of degree $p_a(L) - 1$. It is well-known that the Picard group of $\pi(X) \cong \mathbb{F}_r$ is generated by divisors E_0 and f which satisfy $E_0^2 = -r$, $E_0 \cdot f = 1$ and $f^2 = 0$. Since X is a K3 surface the projection formula yields that L is of the form $L = \pi^*(E_0 + bf)$ for some $b > r$ and the branch locus B satisfies $\mathcal{O}_{\mathbb{F}_r}(B) = \mathcal{O}_{\mathbb{F}_r}(2(2E_0 + (2+r)f))$. Setting $M = n(E_0 + bf)$ we see that M is k-jet ample and

$$\mathcal{O}_{\mathbb{F}_r}(M - \tfrac{1}{2}B) = \mathcal{O}_{\mathbb{F}_r}((n-2)E_0 + (nb - 2 - r)f)$$

is $(k-1)$-jet ample for $n > k+2$. Thus the assumptions of the lemma are satisfied and it yields our assertion. $\square$

The following example shows that the bound in the previous proposition is sharp.

Example 2.3. Let $\pi : X \to \mathbb{P}^2$ be a double cover branched over some smooth sextic $B \subset \mathbb{P}^2$ and let $L = \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$. We are going to show that $(k+1)L$ is not k-jet ample. To this end, choose a point $x \in R = \pi^*(B)_{\mathrm{red}}$ and let u, v be local coordinates around x (as in Case 3 of the proof of Proposition 2.2).

Now consider the jet $J \in H^0(X, \mathcal{O}_X((k+1)L) \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$, which is locally given as

$$J = uv^{k-1}.$$

If $(k+1)L$ were k-jet ample, then there would have to be a section s in $H^0(X, (k+1)L)$ such that $s \bmod \mathfrak{m}_x^{k+1} = J$. Eq. (7) then shows that (locally) s is of the form

$$s = \pi^* s' + u \cdot \pi^* s'',$$

where $s' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k+1))$ and $s'' \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-2))$ are sections such that

$$s'' \bmod \mathfrak{m}_{\pi(x)}^k = v^{k-1},$$

which is certainly impossible.

3. Seshadri constants on K3 surfaces

In this section we explain the relationship between our results and Seshadri constants of line bundles on K3 surfaces.

Recall that the *Seshadri constant* of a nef line bundle L on a smooth projective variety X at a point $x\in X$ is, by definition, the real number

$$\varepsilon(L,x)=\inf_{C\ni x}\frac{L\cdot C}{\mathrm{mult}_x(C)},$$

where the infimum is taken over all irreducible curves $C\subset X$ passing through x (see [2]). The number $\varepsilon(L,x)$ can be thought of as a measure of the local positivity of L at the point x. The infimum

$$\varepsilon(L)=\inf_{x\in X}\varepsilon(L,x)$$

is the *global Seshadri constant* of L.

For K3 surfaces one has the following elementary observation:

Proposition 3.1. *Let X be a K3 surface and let L be an ample line bundle on X. If L is globally generated, then we have*

$$\varepsilon(L)\geq 1.$$

If L is not globally generated, then

$$\varepsilon(L)=\tfrac{1}{2}.$$

Proof. The first part of the proposition holds on any smooth projective variety. It follows from the fact that if $|L|$ is free, then for any given curve $C\subset X$ and any point $x\in C$ there is a divisor in $|L|$ meeting C properly at x.

If L is of the form (1) and $C\subset X$ is singular at x, then either $L\cdot C/\mathrm{mult}_x(C)\geq 1$ or C is a singular fibre of the elliptic fibration, in which case $L\cdot C/\mathrm{mult}_x(C)\geq 1/2$. □

Via vanishing, bounds on $\varepsilon(L,x)$ yield criteria for the generation of jets. If $|L|$ is free, then $\varepsilon(L,x)\geq 1$ implies that $\mathcal{O}_X(nL)$ generates k-jets at x for $n\geq k+2$, and if $|L|$ has base points, then $\varepsilon(L,x)\geq\frac{1}{2}$ implies that $\mathcal{O}_X(nL)$ generates k-jets at x for $n\geq 2k+4$ (see [5, Proposition 5.7]).

Remark 3.2. The proposition is also implied by the theorem stated in the introduction. This follows from the fact that the Seshadri constant of L at x is the relative number of jets that high multiples of L generate at x:

$$\varepsilon(L,x)=\limsup_{n\to\infty}\frac{s(nL,x)}{n},$$

where $s(nL,x)$ is the maximal integer s such that $\mathcal{O}_X(nL)$ generates s-jets at x [2, Theorem 6.4].

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